The Importance of Being Bounded

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It's a matter of mathematical logic. There had been mental illness in my family... - André Bloch (1893-1948).

The old man in the white garments was getting closer but I could not move, not a muscle. I could hear his steps in the living room, slowly approaching the door to my bedroom where, in bed and unable to move, I was hopelessly waiting for him. Then he said it again -*did you prove it yet?*- and a second time, with the scornful tone and the thick german accent that I had heard so many times before. I was terrified but I could not open my mouth. He had come to visit me many times in the past and none of them with good intentions. Finally, he stood in the doorway of the bedroom and said *did you prove it...* and after an uncomfortable pause he said the unavoidable "*yet?*" with a sarcastic smile, almost devilish, on his familiar yet unrecognizable face.

My wife, beside me, was fast asleep. *If only I could wake her up, she would save me*, I thought. So, as on many other nights, I started to do the only thing that I could do in my immobile and desperate state: breathing in a startled, forceful way and increasingly fast. The air rushing out of my lungs was loud enough to wake her up and my wife knew the signal perfectly well by now. As soon as I pulled her out of her slumber, she realized what was happening and proceeded to wake me up and calm me down. *Honey, it's ok, it's ok! It was just a bad dream.*

Indeed it was just another bad dream but the second one that night. Almost immediately after she understood that I was awake and well, she fell back asleep, holding my hand. These nightmares happened so often and her sleep was so sound that she could pacify me in her dreams and never even remember that she did so. I carefully undid her tight grip around my hand, left the bedroom and started pacing up and down the length of the apartment. It was still dark outside but the moon was glowing on the horizon, casting enough light to help me avoid the furniture dispersed around the apartment. I was still sweating profusely and could not stop thinking about the man and his impertinent question. *No, I have not proved it yet, no. There is a lemma missing... it's just a lemma.* In hindsight, it was not *just* a lemma. In fact, it took another three months to prove the "insignificant lemma". *An insignificant lemma*, I repeated in a soft voice. That is how the anonymous referee of a previous version of the paper had referred to the main result in the manuscript, *almost a year ago now? Yes, it has been that long.* Obviously, with such an endorsement, the paper was rejected for publication. One year later and after much work, I had greatly improved my results but I was still torturing myself thinking of my... *should I call it a theorem?...* as an insignificant contribution. It didn't seem that I could ever prove anything that the

referee (who was undoubtedly the old man in the white garments of my nightmares) would consider worthy of publication.

All of a sudden, I realized that I did know the face of the man in my agitated dreams. It was the disapproving look of Kronecker. *How appropriate.* I remembered his round face, white hair and the bulk of his large hooked nose from an old photograph in a biography I had read long ago. In the second half of the XIX century, Leopold Kronecker, a former student of Kummer, Dirichlet and Steiner, became a very influential and internationally respected mathematician, especially in Germany where he co-directed the mathematical seminar of Berlin and was editor of several important journals. However, his views of mathematics were quite strict and somewhat controversial; Kronecker strongly believed that mathematics should deal only with finite numbers and with a finite number of operations and thought of non-constructive mathematics as deceiving and utterly meaningless. In fact, other contemporaries supported these ideas, but Kronecker would prove to be a ferocious defender of his view of how mathematics ought to be. When a young student at Berlin, Georg Cantor, submitted some of his famous ground-breaking results on the concepts of infinity, transfinite numbers and dimension to the 'Journal für die reine und angewandte Mathematik' (widely known as *Crelle*'s journal), Kronecker started a long and harsh campaign to prevent the publication of such work and to discredit Cantor as a mathematician. According to Kronecker, Cantor's work dealt with mathematical objects which could not possibly exist. In fact, although Mittag-Leffler was one of the first mathematicians to accept Cantor's theory of infinite sets, he also tried to persuade Cantor not to publish his results for they were "*about one hundred years too soon*" to be fully accepted by the community. In addition, Cantor despaired for many years trying but failing to prove the so-called continuum hypothesis, which claims that the order of infinity of the real numbers is the next after that of the natural numbers (ironically, in 1963 Gödel and Cohen proved that the truth of the continuum hypothesis was mathematically undecidable, so all of Cantor's efforts were ultimately in vain and doomed to fail). All these developments drove Cantor to a highly depressive and unstable state, and he spent his last years in a sanitorium trying to prove that Francis Bacon wrote Shakespeare's plays.

At any rate, the fierce attitude of Kronecker and Cantor's desperation from repeatedly failing to prove the continuum hypothesis eventually destroyed Cantor's life, and that was a thought that disturbed me. *Could mathematics destroy my own life? I would not let it happen.* Mathematics was my passion and my career but I was not willing to sacrifice years off my life, my family or my friends for mathematics (or so I thought). However, many others, knowingly or unknowingly, have walked down another path, and mathematics did take over their lives in some way or other. J. E. Littlewood admitted that "*Mathematics is a dangerous profession; an appreciable proportion of us go mad.*" Indeed, the list of famous mathematicians that had suffered mental problems was long and sad: Aitken, Baire, Bloch, Boltzmann, Cauchy, Clausen, Eisenstein, Hazlett,... Of course, all sorts of people suffer from psychological problems at some point in their lives, but I always feared that mathematicians were particularly prone to the illness of the mind. *I wonder if these men and women suspected that their dedication to abstract thought was putting them on the edge of madness. I wonder if there were signs. Maybe they recognized them too late. Or perhaps they thought that mathematics would be the cure to their illness.* Alfred North Whitehead, a mathematician and philosopher who collaborated with Bertrand Russell on the famous Principia Mathematica, allegedly said, "*Let us grant that the pursuit of mathematics is*

a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings." However, I thought, the human spirit could be easily corrupted by prolonged abstract thought and the obsession of proving the next big theorem. *It is not a theorem really, perhaps a proposition... or just an insignificant lemma*, I said to myself, thinking about my own research.

I thought of my own mathematical dreams as a worrying sign. Dreams and nightmares with mathematical content were very common, especially in the most productive times. Obsessed about a conjecture, I would work until late and have trouble falling asleep; for I would go through the arguments of a potential proof over and over again only to discover a gap in my reasoning that would bring me to square one. My mother told me many times not to eat right before going to sleep, but nobody advised me against doing math until late. And then, once I was in the arms of Morpheus, my brain would keep working on the problem, drowning in a tide of symbols, equations and diagrams that cluttered my dreams. Sometimes I would not remember the dream itself, but for a second or two after waking up, the torrent of greek letters... $\Omega(\zeta_p)/\Omega(\zeta_p + \zeta_p^{-1})$... would make me dizzy and confused. On occasion, I would wake up realizing that I had made progress in my dreams, and would rush to the computer and LaTeX the proof of the next part of the argument.

I had always been a very active dreamer. Since I was little I would sleep-walk and sleep-talk, and was able to carry out full conversations with my parents that I would never remember upon waking. Although my parents were amused at first, they started to worry sick that I would jump off of the balcony (we lived on the fourth floor) while dreaming that I was a bird or a superhero, which I often did. Hence, they proceeded to install locks on the windows and door to the balcony that they would religiously bolt every night.

The mathematical sleep-walking started with college. I will always remember the first time it happened, in my freshman year, for it was one of the rare times that I actually remembered the dream once I woke up. The undergraduate mathematics program in my university was extremely demanding and the failing rate was excessively high, due to, in my opinion, two factors: the lack of motivation of students who reluctantly ended up there after failing to be admitted to the more popular programs (such as the engineering schools), and the unreasonably high expectations of the faculty. Some of the professors would not hesitate to fail a 90% of the class if they deemed it necessary (and a similar number of students would almost routinely fail some of the toughest courses). In any case, as the first Real Analysis midterm drew closer I was truly frightened and overwhelmed by the material. The pressure could be felt in the air around me. The night before the test I went to bed early and probably started to dream soon after. As strange as it may seem, I was convinced that I was myself a sequence in a metric topological space. Not only was I a sequence, but it was obvious to me that I was bounded since the door and window in my room were closed. Therefore, by the celebrated Bolzano-Weierstrass Theorem, there existed a convergent subsequence of myself. Apparently, after coming to this perplexing realization I looked at the alarm clock next to me: it was about 3 in the morning (the alarm was set for 7AM). However, the fact that a subsequence of myself was convergent implied that I had to get up right then and start making breakfast. Minutes later, I found myself in the kitchen warming up some milk, extremely confused although not completely awake. Somehow, I still believed myself to be a bounded sequence (no windows were open here either) but I started doubting that the Bolzano-Weierstrass theorem necessarily implied that I had to wake up that early. At that point, I went back to bed mumbling *it just doesn't follow!*

Fortunately, in the midterm we were asked to prove the Bolzano-Weierstrass theorem, and, of course, I aced that question, for I myself had once been a bounded sequence.

The bounded sequence anecdote has always been exceedingly popular among my math-friends (my other friends just stare at me, bewildered, when I tell the story) and I have repeated it many times on social occasions. However, as I grew older I started to consider that dream as a premonition of worse things to come in a not so distant future. Several years later, searching in vane for any sort of explanation of my ridiculous dreams, I read the biographies of Bolzano (his full name was Bernard Placidus Johann Nepomuk Bolzano) and Karl Theodor Wilhelm Weierstrass. Bolzano was a modest dealer of art who spent his life fighting with respiratory problems. On the other hand, in [6] one finds the following:

> "From around 1850, Weierstrass began to suffer from attacks of dizziness which were very severe and which ended after about an hour in violent sickness. Frequent attacks over a period of about 12 years made it difficult for him to work and it is thought that these problems may well have been caused by the mental conflicts he had suffered as a student, together with the stress of applying himself to mathematics in every free minute of his time while undertaking the demanding teaching job."

I just could not apply myself to mathematics in every free minute of my life... I would go insane!, I honestly thought. *Did Weierstrass not realize this himself?* Also, I learned that Kronecker furiously opposed the Bolzano-Weierstrass theorem for its non-constructive nature. *No wonder Kronecker doesn't like me*, a bounded sequence said once.

My computer was on the dining table, still on stand-by. The temptation was too great so I gave in and checked my email. Erasing a bunch of spam calmed me down; the ridiculous subject lines of spam emails always cheered me up -*don't be the "small" one of your friends!* Feeling a little better, I went back to bed.

Soon enough, I was back in graduate school, sitting down in the office of my thesis advisor. However, my gentle advisor was not there and Kronecker was sitting behind the desk, staring at me with his stern little eyes. It was an amusing sight this time, for his little legs were almost hanging from the chair. Kronecker was of small stature and was extremely self-conscious about this fact. Once a mathematician by the name of Hermann Schwarz (a student of Weierstrass) dared to write in a letter to Kronecker "*He who does not honour the Smaller, is not worthy of the Greater*," where 'the Greater' referred to Weierstrass, who was a large man. Needless to say, Kronecker did not find this joke amusing at all and never had contact with Schwarz again.

-*Did you prove it yet?* - he asked me, rhetorically.

-*No, Herr Professor, not yet,* - I replied sheepishly - *I am stuck with this... insignificant... lemma.*

-*If it is insignificant, why are you even bothering to prove it in the first place?* - Kronecker replied, frustrated. Herr Professor was playing with his black bow tie and looked annoyed, as if he was losing his time by talking to me. I think he meant to follow up with *Why are you bothering me?* but he refrained.

-*Excuse me, I don't know why I said 'insignificant'* - Liar! I knew perfectly well. - *I do need to prove this lemma as a step towards the main result.*

-*Tell me then, the statement of your claim.*

I stood up, a piece of chalk appeared in my hand and I started writing on the board. For the longest time, Kronecker did not even bother to look at the blackboard, he kept playing with his bow tie (for a minute Kronecker resembled Twiddledee, or maybe Twiddledum, I could never tell them apart) and his eyes were fixed on a bird that was happily singing outside his office. I began:

-*Suppose that* $f(x,y)=0$ *and* $g(x,z)=0$ *are two elliptic curves defined over a number field* K *but sharing the variable* x*. My feeling is that there are at most only finitely many triples* (x,y,z) *in* K^3 *that satisfy both equations.*

-*That is obviously true, unless both curves are isomorphic and of positive rank, of course* - interjected Herr Professor.

I went back to my chair and sank down, embarrassed and looking back at my own writing on the board. The chair was now enormous and my feet were now the ones hanging far from the floor. Actually, the size of the chair had not changed a bit, it was just me, back in early childhood.

-*I am afraid I don't see why, sir.*- I felt about to cry but I contained the tears, out of mathematical pride.

Twiddledum (or maybe Twiddledee) now looked back at me from my advisor's spartan chair, entertained as if this was some sort of game, while playing madly with his bow tie.

-*It is trivial, my child*, Twiddledee (or dum) started saying, *it is just an application of Faltings' theorem, is it not?* - he said in a high pitch voice, almost singing. Curiously, he retained the professor's thick german accent.

I looked down regretting the fact that I came here to ask such an insignificant question. When I looked up, Kronecker was back in the chair, scrutinizing my expression, trying to guess if I had any idea of what he was saying. I knew the words, but the meaning was not at all clear yet. Faltings' theorem (formerly known as Mordell's conjecture) is a remarkable milestone in arithmetic geometry: a curve of genus greater than one can only have finitely many points defined over a number field. The first thing I thought was that Faltings' theorem was not proved until 1983, but Herr Professor Kronecker died almost a century earlier, in 1891. Surely, he would not be amused if I reminded him of the fact that he died so long ago, so I kept quiet. Kronecker was growing more and more impatient in his chair until he jumped on his feet and chalk in hand approached the board.

-*It is a matter of mathematical logic, really. The equations* $f(x,y)=0$ *and* $g(x,z)=0$ *define an algebraic curve* C *in* $\mathbb{P}^2$*, which is equipped with a ramified map down to the elliptic curve* $f(x,y)=0$*. Hence, by the Hurwitz genus formula, the curve* C *has genus greater than one and by Faltings' theorem* C *may only have finitely many points over* K*.*

At that point, I jumped out of bed and ran to the computer which was still on. My wife woke up for a second, enough to rightfully complain -*is everything O.K. honey?*- and then went back to sleep without waiting for an answer. I checked my email -*first things first*- and then LaTeXed the answer that Kronecker had provided in my sleep, almost feeling guilty that I didn't think of this myself. *Good, one step closer to the proof of the... inconsequential proposition.* Once I was satisfied with the writing I stopped to think for a second. "*It is a matter of mathematical logic*," Herr Professor had said. The sentence could not be attributed to Kronecker but to André Bloch. It was still dark out and I needed to get some sleep; the last thing I needed was to start dwelling on Bloch's life again so I decided to pick up a book and read for a while. Even though the book I was reading at the time was "The Feast of the Goat" (a novel by Mario Vargas Llosa about the dictatorship of Trujillo, alias

The Goat, in the Dominican Republic), one of the best and most intense books I've ever read, I could not concentrate and could not stop thinking about André Bloch. *It's a matter of mathematical logic. There had been mental illness in my family...*, I mumbled, repeating Bloch's infamous words.

The first time I heard of Bloch was during my last year in college, in a second course on complex analysis, when the professor proved one of the many pearls that shine in the theory of holomorphic functions (this and other pearls can be found in [5]).

Bloch's Theorem. *Let $\mathcal{F}$ be the set of all functions f holomorphic on a region containing the closure of the disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying $f(0) = 0$ and $f'(0) = 1$. For each $f \in \mathcal{F}$ let $\beta(f)$ be the supremum of all numbers r such that there is a disk $S \subset D$ on which f is injective and $f(S)$ contains a disk of radius r. Let B be the infimum of all $\beta(f)$, for $f \in \mathcal{F}$. Then $B \geq 1/72$.*

The number B is usually referred to as Bloch's constant. Nowadays, better bounds are known and, in fact, it has been conjectured that B has the following tantalizing form

$$B = \frac{\Gamma(1/3) \cdot \Gamma(11/12)}{\left(\sqrt{1+\sqrt{3}}\right) \cdot \Gamma(1/4)}$$

where $\Gamma(x)$ is the ubiquitous Gamma function. To my dismay, the professor made a comment that had a great impact on me: Bloch proved his theorem while living in a psychiatric hospital. After the lecture, I went to the library torn between the intrinsic beauty of the theorem and the fact that Bloch was mentally ill at the time he proved it. *How is it even possible? Are mathematics and madness somewhat compatible?* After some research, I found out quite a bit more about Bloch. For many years to come, I would go back to the almost sensationalist article of Campbell [3], "Beauty and the Beast: The Strange Case of André Bloch", and its encyclopedic counterpart by Cartan and Ferrand [4], "The Case of André Bloch", trying to understand what really happened inside André Bloch's brain.

As Henri Cartan and Jacqueline Ferrand explain in their article, Bloch spent the last 31 years of his life in the psychiatric hospital of Saint-Maurice, near Paris. Astonishingly, he did most (if not all) of his mathematical work during this period, at a small table in a corner of a corridor, refusing to go outside, arguing with the nurses that "*Mathematics is enough for me*," thank you. The nurses confirmed that André was an "ideal patient," quiet and extremely polite during his life spent in the "House of Health," as Saint-Maurice was often called. Bloch maintained a lively communication with several mathematicians: Valiron, Cartan, Hadamard, Pólya, Baidoff, Picard, Montel and Mittag-Leffler among others. Some never suspected that the sender's address 57 Grande Rue, Saint-Maurice, corresponded to a sanitorium; others were in shock when they tried to pay a surprise visit to their colleague. Many found puzzling that Bloch insisted on dating most of his letters from April 1st.

When I found out the ultimate reason why André was a permanent resident in the House of Health, I was so distraught that, when I went to bed that night, I had many dreams and nightmares about Bloch, holomorphic functions, Landau's constant L and discs of radius B that flew around my head. In one of my dreams, I was visiting André at Saint-Maurice. The nurses took me to a long corridor and pointed at the very end, where a man was bent over a desk. After walking what seemed to be miles, and after passing by some other patients (Randle P. McMurphy was whispering confident words of freedom to the silent

giant "Chief" Bromden; Salvador Dalí was painting what seemed to be an elaborate Greek letter zeta ζ devouring a summation sign $\Sigma_{n=1}^{\infty}$ which in turn was jealously protecting the summand $1/n^s$ under its wings), I reached the end of the corridor and had a close look at Bloch. In my dream, I was not surprised to see that Bloch was in full military uniform, although I was shocked to see that the nurses would allow a patient to have a cavalry sword on his belt. In 1914, only one year after André and his younger brother Georges (also very talented at mathematics) had entered the Ecole Polytechnique of Paris, they were both drafted into the army, as the first World War had begun, shaking all of Europe. The corridor had large windows, overlooking magnificent gardens that never seemed to end, those magnificent gardens that Bloch did not seem to care about. Outside, sitting on a bench, Georges was enjoying the Sun, also wearing a military outfit. From time to time he would adjust his eye-patch which seemed loose around the head. Both brothers had been badly wounded during the war: Georges had lost an eye; during a bombardment André fell from an observation post which left him in a grave state and some speculate that the fall may have triggered his mental problems.

-*Monsieur Bloch?*- I tried to call his attention but he was too busy drawing circles in the complex plane. -*Monsieur Bloch?*- I repeated in a soft voice. My french is terrible so I was afraid that he couldn't even understand the word *monsieur*.

Bloch turned around and looked at me. His face was calm and a relaxed smile prevailed in his otherwise sad expression. I have never been able to find a picture of Bloch, but his face was vaguely familiar. Later I concluded that I had substituted for André's face that of Alexander Aitken, a mathematician who also fought in World War I. In fact, it seems that the lives of both mathematicians are blended in mind.

Aitken was born in New Zealand, he enlisted the army and fought in the fronts of Gallipoli, Egypt and France. In 1916, badly wounded in the impossible Battle of Somme (which claimed the lives of more than 800,000 men), he was taken back to New Zealand but the war never ended for Aitken. The vivid memories haunted him for the rest of his life, probably causing most of his mental problems and ill health. In vain, he tried to pacify the demons by writing [1], a book about the war. However, as a positive result of his book, he was elected a member of the Royal Society of Literature in 1964.

Bloch was now curiously studying my own appearance. A couple of doctors walked by us, frowning disapprovingly at me. I immediately realized two things: at least one of the doctors was Kronecker (if not both!), in his white robe, and I was still wearing my pajamas -*oh no, not again*-, which made me feel extremely uncomfortable and out of place (just in case being in a psychiatric institution was not unsettling enough). I overheard Kronecker complain to himself about the non-constructiveness of Bloch's theorem. "*Indeed, it is nonsense: the work of a true mad man*," the other Kronecker agreed. André lowered his head, embarrassed.

-*Don't listen to him, everyone praises your work nowadays* - I said trying to comfort him. In fact, the Académie des Sciences awarded him the Becquerel Prize in December of 1948 (two months after his death).

-*And what matter brings you to 57 Grande Rue, Saint-Maurice?* - enquired Bloch, politely. His peaceful eyes irradiated an exuberant curiosity. The expression on his face (Aitken's actually) was kind and encouraging.

-*I came to ask you about... the incident. I must know. Why did you do it?* - My voice was nervously shaking.

-*The incident? What incident do you refer to? Did I do anything wrong?* - Of course, he knew exactly what I was referring to but he could not find fault in his actions. His magnificent memory was intact (once I read that Aitken, who I kept identifying with Bloch in my dream, could recite the first 2000 digits of π). I would have preferred not to pronounce the words, but I had to:

-*Why did you kill them?*

Bloch's younger brother, Georges, was now standing next to me with his eyes fixed on André, also demanding an answer for the committed crimes. Bloch spent most of 1917 in a hospital, recovering from his war wounds. Shortly after his convalescence, on November 17 and during a family dinner, André killed his brother Georges, his uncle and his aunt. There are not many details available, for it was a time of war and such an incident was not the kind of news that would help in the recruiting of new soldiers. According to some unlikely rumors, André killed them all with his cavalry sword. Others claim the weapon was an axe. Most likely, it was some sort of kitchen knife.

Bloch, looking back at his brother's eyes, started talking in a matter-of-fact but gentle voice: *It's a matter of mathematical logic...* Unfortunately, I knew the words perfectly well. Almost at the end of Bloch's life, a family member came to visit him at the House of Health. André asked many questions about the surviving members of the family and their whereabouts. After the visitor left, a doctor asked him about his relatives and the murders and Bloch said:

-*It's a matter of mathematical logic. There had been mental illness in my family. Above all there is mathematics and its laws. You know very well that my philosophy is inspired by pragmatism and absolute rationality. The destruction of the whole branch had to follow as a matter of course. I began my work at the time of that famous meal. It is not finished yet. I wanted to know how things were.*

I slowly opened my eyes and spent the rest of the night staring at the ceiling, deep in thought and overwhelmed by infinite sadness, blaming the madness of War for the loss of millions of brilliant minds.

My wife found me in the living room, reading the words of Bloch that had been already memorized long ago.

-*What's up with you tonight?* - she said. At the same time, she took the papers from my hand, threw them on the floor and dragged me back to the bedroom. Before I could apologize she was dozing off again. I tried closing my eyes but for a while I could only think of those whose ill fate put them in the destructive path of one of the many unjustifiable and unexcusable wars: from Archimedes (287-212 BC), who was killed during the Second Punic War when the Romans took over the city of Syracuse (despite the war machines that Archimedes had invented to defend it), to the millions of people whose lives were shattered by the World Wars. Among them, the extremely active community of mathematicians which was thriving in Germany and France in the first half of the XX century had to either join the war efforts (e.g. Turing perfected the 'Bombe' machine to decipher the codes sent by the machine 'Enigma' of the Luftwaffe) or flee to safer parts of the world (Emil Artin "with his feeling for individual freedom, his sense of justice, his abhorrence of physical violence," as Brauer wrote in [2], had to leave Germany for the safety of his Jewish wife). Others, either stubbornly or rightfully so, refused to leave their home and their universities. When Edmund Landau was warned that, as a result of his Jewish roots, if he was to remain

in Göttingen he would be sent to a concentration camp, Landau (who was known to be arrogant at times) replied that "*In that case I should immediately reserve for myself a room with a balcony and a southern exposure.*" Sadly, it was another renowned mathematician, Paul Julius Oswald Teichmüller, a distinguished member of the rising Nazi party, who led the student protests and succeeded to dismiss Landau from his position at Göttingen.

Felix Hausdorff and his family (also Jewish) decided to stay in Bonn; they thought that the support of Bonn University would be enough to protect them. However, when they learned that soon they would be sent to the concentration camp of Endenich, Felix, his wife and his wife's sister decided to take their own lives first, before losing their dignity as human beings in the hands of Nazis. His last recorded words were for a friend, in a letter sent on January 25, 1942:

> *Dear Friend Wollstein,*
>
> *By the time you receive these lines, we three will have solved the problem in another way - in the way which you have continually attempted to dissuade us. What has been done against the Jews in recent months arouses well-founded anxiety that we will no longer be allowed to experience a bearable situation. Forgive us, that we still cause you trouble beyond death; I am convinced that you will do what you are able to do (and which perhaps is not very much). Forgive us also our desertion! We wish you and all our friends will experience better times.*
>
> *Yours faithfully, Felix Hausdorff.*

I silently wished we would all experience better times than Hausdorff did but I was certain that War and irrationality would come again and again, to absurdly claim the lives and minds of those who stood in their way.

I don't recall how long it took me to fall asleep but I do remember waking up, in the highly drowsy state of half-consciousness which always preceded the sleep-walking episodes. I don't remember putting any shoes on, but I definitely recall walking down the stairs to the first floor, opening the front door and leaving the house, fulfilling my parents second worst nightmare. In my last half-lucid time of the night, I was crossing one of the many bridges of the town (I don't remember which), high above a gorge that was roaring in the otherwise quiet summer night. After that, I went into a deep sleep and I only remember parts of my dreams and not my real-life acts. On the other side of the bridge there was an old brick house. I approached the front door, cleared my throat, and rang the bell twice. A chambermaid greeted me, said *he is almost ready, he will be here shortly*, and invited me to sit down while I waited. Eventually, an old man with a fine white scarf tightly wrapped around his neck, a puffy black hat and a cane descended the stairs, with the help of the maid. I stood up and the maid trusted the arm of the elder to me and escorted us to the entrance, gently closing the door as we stepped out.

Although the man was obviously blind and (I estimated) about 70 years old, he was moving at a brisk pace and his certain steps did not correspond with his lack of vision. It was clear that he had walked the same path many times before; he knew the exact position of every single lamp-post, fire hydrant, corner and pothole on the street. In fact, my feet tripped a couple of times and it was he who prevented my fall, making me reconsider who

was blind and who was leading who. As we walked (a little faster and we would be jogging) I started the conversation. I wasn't sure how to address him, so I decided that 'Master' would suffice:

-*Master, do you think mathematicians are prone to... mental illness?* - I intentionally avoided the word madness.

-*Yes and no. Not any more than any other person who industriously conducts research or is devoted to the arts. In many circumstances the virtues of persistence and dedication may become the sin of obsession, which in turn can lead to frustration, depression and worse conditions of the mind. Pick a discipline and I can provide a long list of examples of well-known figures in the field who suffered deep psychological problems.*

I accepted the challenge, said -*Literature?*- and the Master replied:

-*Hans Christian Andersen, Charles Dickens, Emily Dickinson, F. Scott Fitzgerald, Henrik Ibsen (a Norwegian playwright, considered one of "the great four"),... Shall I go on?* - I shook my head asking him to stop. He continued. - *Edgar Allen Poe, Leo Tolstoy, Virginia Woolf,... Actually, the men of letters and the men of numbers share a common problem: our passion can be carried out in our head, without the need of tools or laboratories. A writer, a playwright or a mathematician must exercise the highest degree of self-restraint so that the brain is not constantly and exclusively dedicated to the creation of a new character in a novel or the proof of an elusive conjecture. Paul Auster wrote in his short story "Ghosts"...*- I was amazed that the Master had even managed to read XX century authors -*... that "writing is a solitary business. It takes over your life. In some sense, a writer has no life of his own. Even when he's there, he's not really there." The same may be said about mathematicians and mathematics.*

As we were chatting, with my right arm intertwined with his left one, we crossed another bridge. Our stroll, apparently, was occurring in the city of Kaliningrad (formerly known as Königsberg) where, of the seven famous old bridges (at least in mathematical circles), only five remained intact after the bombardments of World War II. However, the new configuration was much more mathematically pleasant to the Master, because nowadays it is possible to take a walk across all five bridges over the river Pregel without having to cross any bridge more than once.

-*Did you know that Henri Poincaré would only do mathematical research precisely from 10 to noon and from 5 to 7 everyday?*

-*I did not know that, sir* - I said surprised. I don't know why I acted surprised, because I did know this, of course. After all, you cannot learn new historical facts in your own dreams.

-*Poincaré studied in detail the mathematical and scientific method (read, for example, "Science and Hypothesis," "The Value of Science," and "Science and Method"). He asked himself many times the same questions you bring with you today, and he concluded that it was likely that mathematics could consume his soul. Therefore he took careful steps to prevent it.*

-*Didn't Weierstrass reach the same conclusions? Didn't Bloch ask himself the same questions?*

-*They probably did but they also concluded that their virtue was mathematics and they devoted every minute of their free time, for the advancement of the Queen of Science* - the Master explained.- *It is true, however, that many have overestimated their mental capabilities, thinking that they could dedicate themselves to mathematics as intensely as they*

pleased, without paying the consequences... did you know that Girolamo Cardano claimed to have mathematically predicted the exact date of his own death?

-And was he correct?

-Of course he was, a mathematical proof is infallible! He committed suicide on that exact date.

I have always asked myself to what extremes a man must be disturbed to commit suicide; what an absurd way to depart. Some historians claim that Pythagoras committed suicide after a vicious attack that destroyed the Pythagorean Society that he had laboriously created. Ludwig Boltzmann hanged himself, perhaps because his ideas and differential equations modelling the atomic structure were not accepted by the community. We will never know the reason behind Yutaka Taniyama's suicide, a young man with a seemingly brilliant future. In a note, he wrote "*Until yesterday I had no definite intention of killing myself... I don't quite understand it myself, but it is not the result of a particular incident, nor of a specific matter.*" His fiance killed herself shortly after. Some years ago, a young mathematician I knew announced solely to me that he would kill himself at the end of the semester because he could not find a tenure track position. To my dismay, I also found out that he had attempted suicide in the past. Luckily, after psychiatric intervention, he decided it was not the best idea at this point.

-Don't distract yourself with these sad stories and mere anecdotes. Most mathematicians have had pleasant and extremely fulfilling lives! - the Master reprimanded me. - *On the contrary to your fears, Mathematics has provided for many of us the most joyous and rewarding moments of our lifetime.*

During my academic years I have met many members of the mathematical community and, indeed, most mathematicians seem to live jubilant lives. I can't say that the mathematicians I have met are ordinary people because, in many cases, they are far from ordinary. Nowadays, the mathematical community, of which I am a proud member, is composed of a set of very friendly people, surprisingly down-to-Earth, who love informal settings (e.g. dining and wining) where they can discuss their surprising new counter-examples and the brand new departmental gossip. Although some members of the community exhibit characteristic extravagant behaviors (in some circles being extravagant seems to be encouraged), and even though some members could turn down their mathematical arrogance a notch or two, I must say that most mathematicians are of a fine nature, sprinkled with some inoffensive quirks and mild eccentricity (say mathematical sleep-walking) which in my opinion only adds to the value of the whole. Perhaps, on occasion, we may be socially inept, creating deliciously awkward situations around us. On the other hand, we are able to organize stupendous parties, and even conferences, to celebrate our favorite mathematician's birthday or the discovery of some long sought proof.

The Master and I had stopped on one of the bridges and were now contemplating the Pregel, running downstream in its desperate search for salt. While I was using my eyes, the Master seemed to use every other sense, particularly searching through his own past memories of the river and the city. Despite being totally blind by the time he was 59, he produced half of his mathematical work after that age, due to his prodigious memory which permitted him to keep working every day of the rest of his life, perhaps seeing more clearly than any of us. The Master spoke:

-Think of the most beautiful theorem you know, think of its proof, can you see it at a glance?

I thought of Euclid's proof of the infinitude of the primes; short, simple but absolutely breath-taking in its depth.

Proposition 20 (Euclid's Elements, Book IX)**.** *There exist infinitely many prime numbers.*

Proof. Suppose for a contradiction that there exist only finitely many primes $p_1, p_2, \ldots, p_n$, where p_n is the largest of them. Let $\mathcal{N}$ be defined by:

$$\mathcal{N} = (p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1.$$

Clearly, $\mathcal{N}$ is larger than p_n so it cannot be prime. Hence it must have a prime divisor in the list $p_1, \ldots, p_n$. However, if some prime number p_i divides $\mathcal{N}$, it must also divide $\mathcal{N} - (p_1 \cdot p_2 \cdot \ldots \cdot p_n)$ since both numbers are multiples of p_i. Therefore p_i divides 1, which is impossible. Thus our first assumption must be false and there exist infinitely many prime numbers. □

-That is the most beautiful view you will ever see. Aren't you glad to be a mathematician so that you are able to enjoy these treasures which seem so elusive and out of reach for the mathematically illiterate? The enchanting beauty of a theorem and the fabulous satisfaction of finding a proof is the most worthwhile and fulfilling experience that a mathematician can afford.- The Master concluded, triumphant. After that, in silence and holding my arm, he led the way through the streets of Kaliningrad, back to my house. With a bow, he said goodbye and left, almost running, in the direction of his own house.

I woke up inside the apartment, standing in the doorway to my bedroom. Before recovering full consciousness I looked at my white pajamas, understanding that I was and had always been my own Kronecker, perhaps my worst Kronecker. Even I was surprised when I heard myself saying - *Did you prove it yet?* But I decided I would not torture myself anymore, that I would enjoy what I do regardless of what others (Kronecker or the referee) may think. *It is not an insignificant lemma*, I said to myself, *it is worthwhile as long as I see beauty in it* and the proposition I was trying to prove definitely had some intrinsic beauty. On the other hand, looking at my wife, my nightly savior still placidly asleep, I noticed that Euler, like all of us, also made mistakes: there were more beautiful and fulfilling views than a lemma, a theorem or even Proposition 20 of Book IX.

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