Defending Reasonableness
Division of Fractions

TEACHER: What would this one be? Without doing a rule that you know, like, see if you can make sense of why, what one divided by two-thirds is.
STUDENTS: (*in small group*) It's one and a half . . .
TEACHER: Yeah, but why does it make sense, visually? . . . (*Addressing the whole class*) All right, let's hear what some people have to say. Leslie.
LESLIE: What is the answer?
TEACHER: Yeah, say what you think it is.
LESLIE: I think it's six.
TEACHER: And you think it's six because . . .
LESLIE: Because a third goes into one three times and then two times three is six.
TEACHER: All right. Um, other thoughts about this? Um, Claire.
CLAIRE: I got one and a half. I switched around the three and the two.
TEACHER: I can't hear.
CLAIRE: Um, I got one and a half because I made the three the numerator and the two the denominator . . .
TEACHER: You used the reciprocal of three-halves of two-thirds.
CLAIRE: Yeah.
TEACHER: Why?
CLAIRE: Because on the other one we switched around the numerator and denominator . . .
TEACHER: Yeah, and so see what's, here's what, here's where I'm going with this today. Before the end of class today you're going to have a rule and some people already know the rule, but I don't care about the rule right now. What I want to know is, can anybody make sense of this? Let's just, so right now we have two different answers. We have the answer six and Leslie's saying it's because you divide, there's three one-thirds in one and then two times three is six. And then we have the other theory that the . . . Put your hands down . . . The other theory is that the answer's one and a half, but we don't know why except the rule works. So the rule working is not good enough today. All right, so why does it make sense? Why does Leslie's answer make sense? Why

does one and a half make sense? I want to know why it makes sense . . . OK. I've got almost one hand up in every group but not quite. I want every group to come up with what makes sense. Some reason, why. Put those heads together.

JESSE: (*In small group*) It's like, uh, how many two-thirds are in one. Two-thirds, so that's one, and then half of two-thirds is one-third, two-thirds plus one-third is three-thirds, which is one, right? That's what I thought!

. . .

TEACHER: Excuse me. This is really interesting. The side of the room that I talked to, about half of the people thought it was six and about half of the people thought it was one and a half. And then I found Sam willing to make an argument for one and a half, so, Sam, would you go to the board and draw your picture and you can erase the other pictures so far. And would you erase the other arithmetic, too, so we start with a new problem?

SAM: Well, if you have, if you have this . . . OK, that's not equal but, if that's one whole and if these two parts are shaded, then that's two-thirds, right? Of that. And this one doesn't count now, so that's two-thirds and this is the other one-third and so if you draw . . . this is a whole. See, that's the whole, right? And if you take these two and put them here, like it was, and then you still have these, then you still have the one-third left and one-third is half of two-thirds and so you just add one-half. And that's what I was thinking.

TEACHER: Now, remember the day we talked about convince yourself, convince a friend, and convince a skeptic. Anybody want to, let's see if anybody can challenge Sam or push on his thinking a little bit.

MICHAEL: OK. Well, that's exactly what I was thinking and that's how I convinced our table because we had a couple people here who thought it was six. But then I pretty much convinced them that it wasn't.

TEACHER: Zach or Ben, are you willing to talk about what Michael Ann said that convinced you?

BEN: Everything.

ZACH: I thought it was one and a half at the beginning. He was the skeptic.

TEACHER: You were the skeptic? Now really, this would be really great. What made you change your mind?

BEN: Um, it made more sense to me than before.

TEACHER: Can you explain it now?

BEN: I think so.

TEACHER: OK. Would you try, please? Because if you can explain what somebody else said then you are one notch further up in your ability to understand it. OK.

BEN: Well, it's two-thirds and two-thirds goes into one, one whole time. And there's still a little half left over, or a little one-third left over. And so you try to fit it in there. Not all of it fits, only half of it fits and so it's one and one-half times.
TEACHER: Anyone want to comment to Ben?
EVAN: I agree with you and Sammy and Michael, I just thought of it in a line way. Where it's on a line.
TEACHER: Could you show your line way, then? So, the more ways we have it . . . We're going to have to hear from the six people then because there's got to be maybe some people who are still thinking the answer is six.
EVAN: . . . one and so right here there's a two-thirds mark. So, from like here to here it goes into it one time, one full time, like Ben said. So that's like considered one. Then here is one-third left and one-third is one-half of two-thirds, so times two-thirds equals . . . oh wait. One-third is one-half of two-thirds.
TEACHER: How would you write that? One-third is half of two-thirds?
EVAN: Oh, I think . . . no wait. I'm not sure.
TEACHER: Maybe you could get help from someone in the class.
EVAN: All right, Cheryl.
CHERYL: Two-thirds, you write two-thirds times one-half and then you get one-third.
EVAN: OK, yeah. So then two-sixths is equal to one-third and that's that part. One-third. So that one-third is like one-half of the two-thirds, so then you can consider it one and a half.
TEACHER: Cheryl.
CHERYL: I had another way of thinking about it, like as in a number sentence way.
TEACHER: Will you go up?
CHERYL: OK. So, the problem is one divided by two-thirds and that equals blank. And then also regarding, you know how you can go the inverse, so it can go either way. So then you can do this times two-thirds equals one. And so, that's going to be three over two, which is also one and a half.
TEACHER: Why does that make sense? Or does it make sense?
CHRISTINE: I don't really understand, like, I still think it's six because, um, I . . .
TEACHER: Oh wait, Christine. And I know you've had your hand up and I want to call on you about why it's six but just, what method is Cheryl using here for this one? . . . OK. Since people aren't remembering, remember yesterday we talked about that fam . . . , the four related number sentences?

Can anybody, well, OK, so you chose another number sentence to show why that would be true. OK. Um, Christine, and you still think it's six?
CHRISTINE: Yeah because me and Alicia kind of have the same method. Well, what we did . . .
TEACHER: You and who have the same method?
CHRISTINE: Alicia.
TEACHER: OK.
CHRISTINE: And like well, we, multiplied the numerator by the denominator and then multiplied that by the whole number, which equals, because two times three equals six.
TEACHER: Could you go up and show what you did? . . . Is this the kind of problem that you can have two different answers to?
STUDENTS: No.
TEACHER: No. Some problems could have more than one answer or a range of answers, but in this case we agree that there has to be, so the question is which is the most sensible way of interpreting this problem?
CHRISTINE: Um, two times three equals six and um, six times one equals six.
TEACHER: And why does that make sense, Christine?
CHRISTINE: I don't . . . um . . .
TEACHER: Leslie, what do you think about, what do you think about your original answer?
LESLIE: Well, I thought it was six but then I don't think it can be that big.
TEACHER: Ah, why don't you think it could be that big?
LESLIE: Well, because you're doing, like, one divided by two-thirds, and it would be less than six. I don't know any other way to do it.
TEACHER: So you were trying to figure out, and you're trying to figure out, something to do with numbers.
CHRISTINE: I was thinking about Cheryl's method and I don't really understand it that much, but I'm trying to, I was looking at it and it makes sense now.
TEACHER: OK, we've got two people who are just about having a fit over here. Would you call on a few people?
CHRISTINE: Um, Amy.
AMY: Well, if you multiply six by two-thirds, it wouldn't equal one.
TEACHER: Oh, you're checking the inverse operation.
AMY: Yeah.
CHERYL: You know you wrote two times three. That wouldn't be right because the line um, is, means divide so it would be two divided by three, kind of-ish. Because the line separating the two and the three is a divide sign.
TEACHER: OK. So I'm going to stop this right now because here's something that really, I think is the biggest danger in math is that you follow rules, you do things with numbers. It's kind of like waving your magic wand. As a mat-

ter of fact, I have a cartoon like that. Where this mathematician is doing something on the board and there's all these things that he's done and then in the middle, let's see, how does it go? In the middle there's this thing where you kind of wave your magic wand and that way you prove your answer. So, that's what I don't want you to do. I want you to make sense of it. And I want to go back to what Leslie said about how six is too big to be sensible. So, let's see, what would be a situation where this would make sense? Um, two-thirds is kind of hard, let's see. Um, I have a yard, oh gosh, let's see. I have a foot of lumber, and I need to cut off two-thirds of a foot, I need to cut my lumber into two-thirds-of-a-foot sections. So how many two-thirds-of-a-foot sections will I be able to get out of there? So if you visualize that piece of lumber, I cut off the two-thirds of a foot, how much I've got left . . . is that, does six make sense then?

STUDENTS: No.

TEACHER: No. It couldn't be six. It's too . . . so I really like the way, Leslie, you were reasoning about that. Um, what would it be then? How many pieces of lumber could, how many two-thirds-of-a-foot sections could I get? Leslie?

LESLIE: I think one and a half.

TEACHER: OK. And why do you think one and a half?

LESLIE: Um, because it's a foot and you're taking two-thirds . . .

TEACHER: I've got a foot of lumber. I should have said, you don't usually have lumber in one-foot sections, but anyway . . .

LESLIE: So, there's two-thirds and there's one-third left and one-third's half of two-thirds.

TEACHER: OK, so, if I have a, here's my foot of lumber right here. And I cut off my, I cut off my two-thirds and so I get rid of that, and then I've only got this little piece left. So I've only got one two-thirds-of-a-foot section, this one. And then this part is how long?

STUDENTS: One-third of a foot.

TEACHER: One-third of a foot, and that's half of what I need. So one and a half is the correct answer. Now, Christine, does that make any sense to you?

CHRISTINE: Yeah, well, it does make sense. Like I, when I went to the board I kind of was thinking about it, like I thought maybe Leslie was right and it was too big. But like I was, like, I know it's just probably confusing but wouldn't it, like, the foot of lumber like divided by three, twelve divided by three is four.

TEACHER: Where are you getting twelve?

CHRISTINE: Because, twelve inches in a foot.

TEACHER: Oh, so you're thinking twelve inches in a foot. Yes, and so each third would be four inches. I agree with that. And two-thirds of a foot is how many inches?

CHRISTINE: Eight inches.
TEACHER: Eight inches. So I cut off eight inches and I've got how many inches left?
STUDENTS: Four.
TEACHER: Four. And four is half of eight. So that's another way to prove it. Good. OK. Now, I'd like you to copy this down, because this is called the traditional algorithm for dividing fractions. There's more than one way to divide fractions, but this is the traditional algorithm . . .