# Sequences and Series

Quantitative Learning Center at the University of Connecticut

## Sequences

A *sequence* is a list of numbers

$$\{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

The sequence *converges* if

$$\lim_{n\to\infty} a_n$$

exists. If the limit does not exist, then the sequence *diverges*.

## Series

A *series* $\sum_{n=1}^{\infty} a_n$ is the limit of the *partial sums*

$$s_n = \sum_{i=1}^{n} a_i.$$

If $\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_1 + \cdots + a_n$ exists, then the series *converges*. Otherwise the series *diverges*.

**Bookkeeping on indices**:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} a_{n-1} = \sum_{n=0}^{\infty} a_{n+1}.$$

Write a few terms to see why.

### Special Cases

#### Geometric Series

When $|r| < 1$,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

When $|r| \geq 1$, this *geometric series* diverges.

#### Telescoping Series

The *telescoping sum*

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} \\ &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots \\ &\quad + \tfrac{1}{n-1} - \tfrac{1}{n} + \tfrac{1}{n} - \tfrac{1}{n+1} \\ &= 1 - \tfrac{1}{n+1} \end{aligned}$$

has the limit

$$\lim_{n\to\infty} s_n = 1.$$

#### $p$-Series

The *p-series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

- converges if $p > 1$ and
- diverges if $p \leq 1$.

This follows from the integral test.

#### Example: Harmonic Series

The *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is a $p$-series with $p = 1$, so it diverges.

### Test for Divergence

Given a series

$$\sum_{n=1}^{\infty} a_n,$$

if

$$\lim_{n\to\infty} a_n \neq 0$$

or doesn't exist, then the series diverges (e.g., $\sum n/(n+1)$ diverges). If

$$\lim_{n\to\infty} a_n = 0,$$

then you can't draw any conclusions (e.g., $\sum_{n=1}^{\infty} 1/n$ diverges but $\sum_{n=1}^{\infty} 1/n^2$ converges).

## Integral Test

Suppose that there is a function $f(x)$ such that $f(n) = a_n$ and suppose $f(x)$ satisfies three conditions for some $c$:

1) $f(x)$ is positive on $[c, \infty)$.
2) $f(x)$ is continuous on $[c, \infty)$.
3) $f(x)$ is decreasing on $[N, \infty)$ for some $N \geq c$.

If (1), (2), and (3) are met, and

$$\int_c^{\infty} f(x)\, dx$$

converges, then

$$\sum_{n=c}^{\infty} f(n) = \sum_{n=c}^{\infty} a_n$$

converges.

If (1), (2), and (3) are met, and

$$\int_c^{\infty} f(x)\, dx$$

diverges, then

$$\sum_{n=c}^{\infty} f(n) = \sum_{n=c}^{\infty} a_n$$

diverges.

*Note:* The integral test is often used with $c = 1$.

### Estimating Sums

Suppose the series

$$s = \sum_{n=1}^{\infty} a_n$$

converges, but you don't know its value. You can estimate its value with a partial sum, $s_n$. The total sum will be equal to $s_n$ plus a remainer, $R_n$:

$$s = s_n + R_n.$$

If $a_n = f(n)$ as in the integral test, and (1), (2), and (3) are satisfied, then the remainder $R_n$ is bounded by

$$R_n \leq \int_n^{\infty} f(x)\, dx$$

for $n > c$.

## Comparison Test

Suppose all $a_n$ in one sequence are positive and all $b_n$ in another sequence are also positive.

### Direct Comparison

- If $0 < a_n \leq b_n$ and
$$\sum_{n=1}^{\infty} b_n$$
converges, then
$$\sum_{n=1}^{\infty} a_n$$
converges and $\sum a_n \leq \sum b_n$.
- If $0 < a_n \leq b_n$ and
$$\sum_{n=1}^{\infty} a_n$$
diverges, then
$$\sum_{n=1}^{\infty} b_n$$
diverges.

### Limit Comparison Test

If $a_n > 0$ and $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

with $0 < c < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. That is, if one converges then so does the other, and if one diverges then so does the other.

## Alternating Series

*Alternating series* look like

$$\sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + \cdots$$

or

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - \cdots$$

where all $b_n > 0$. Example: the *alternating harmonic series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$$

### Alternating Series Test

This series converges if

- $b_{n+1} \leq b_n$ for all $n$ and
- $\lim_{n\to\infty} b_n = 0$.

### Estimation Theorem

If

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n, \quad b_n \geq 0,$$

converges, and $b_{n+1} \leq b_n$ for all $n$, you can estimate its value

$$s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

by taking the partial sum

$$s_n = \sum_{i=1}^{n} (-1)^{i-1} b_i.$$

The error in this estimate is at most the magnitude of the first omitted term, $b_{n+1}$:

$$|R_n| = |s - s_n| \leq b_{n+1}.$$

*Note:* this also works for alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^n b_n.$$

## Absolute Convergence, Root, and Ratio Tests

### Absolute Convergence

A series $\sum_{n=1}^{\infty} a_n$ *converges absolutely* if

$$\sum_{n=1}^{\infty} |a_n|$$

converges. Every absolutely convergent series is convergent. The series *converges conditionally* if the series $\sum_{n=1}^{\infty} a_n$ converges but

$$\sum_{n=1}^{\infty} |a_n|$$

diverges (e.g., alternating harmonic series).

### Ratio Test

- If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$$
then $\sum_{n=1}^{\infty} a_n$ converges absolutely, so the series converges. Example: $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges ($L = 0$).
- If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$$
or
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$$
then $\sum_{n=1}^{\infty} a_n$ diverges.
- If
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$$
then you can draw no conclusions (e.g., $\sum 1/n$ diverges, $\sum 1/n^2$ converges).

### Root Test

- If
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$$
then $\sum_{n=1}^{\infty} a_n$ converges absolutely, so the series converges.
- If
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$$
or
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$$
then $\sum_{n=1}^{\infty} a_n$ diverges.
- If
$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$$
then you can draw no conclusions (e.g., $\sum 1/n$ diverges, $\sum 1/n^2$ converges).

## Convergence Test Strategies

Classify the form of the series.

1. If $a_n \not\to 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.
2. If the series is of the form
$$\sum_{n=1}^{\infty} ar^{n-1} \quad \text{or} \quad \sum_{n=0}^{\infty} ar^n$$
then use the geometric series.
3. If the series is $\sum_{n=1}^{\infty} \frac{1}{n^p}$, then use what you know about $p$-series.
4. If $a_n = f(n)$ and $f(x)$ is easy to integrate, try integral test.
5. If the series has either form
$$\sum_{n=1}^{\infty} (-1)^n b_n \text{ or } \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$
with $b_n > 0$ and decreasing, try the alternating series test.
6. If $a_n$ grows like $b_n$ and $\sum_{n=1}^{\infty} b_n$ is known, use limit comparison test (positive terms).
7. If $|a_n| \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, use comparison and absolute convergence.
8. If the series has $n!$ or $n$th powers, try the ratio or root tests.