Sequences and Series Review

Math 1132Q/1152Q

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What is a sequence?

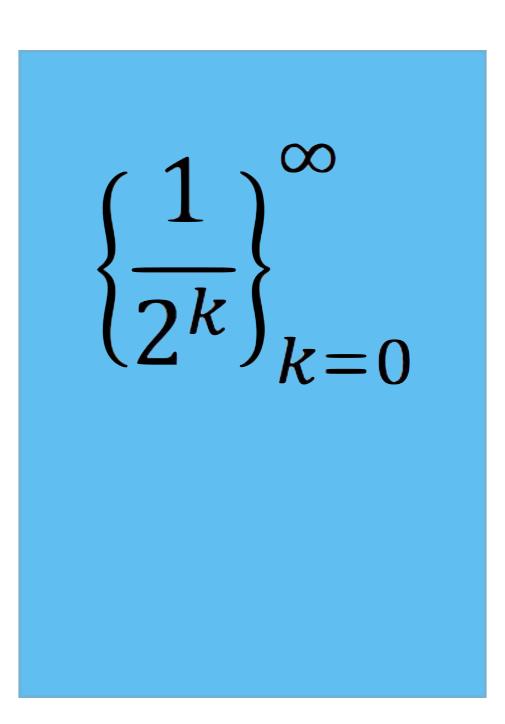
A sequence is a list of numbers with a specified order.

Ex. $\{a_1, a_2, a_3, \ldots, a_n\}$

The sequence converges if:

 $\lim_{n \to \infty} a_n \qquad \text{exists.}$

Otherwise, it diverges.



What is a series?

A series is the sum of an infinite sequence.

$$\sum_{n=1}^{\infty} a_n$$

A partial sum is the sum:

$$\sum_{i=1}^{n} a_i = s_n$$

lf:

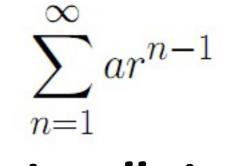
 $\lim_{n\to\infty} s_n \quad \text{exists,}$

then, the series converges.

Otherwise, it diverges.

Geometric Series

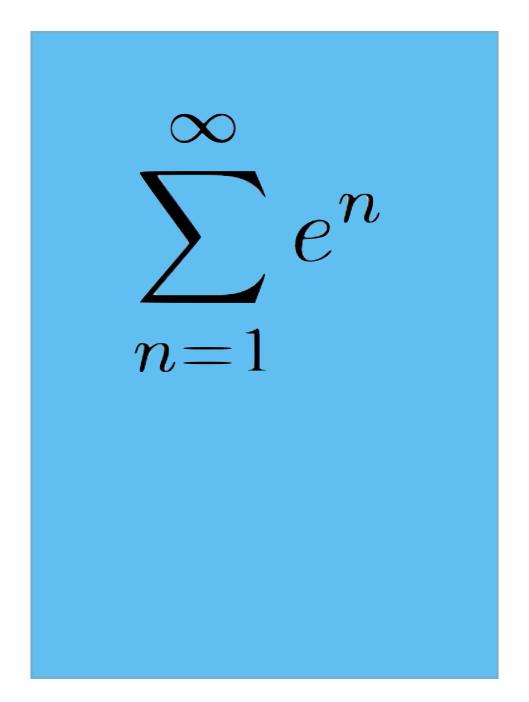
When we have a series of the form:



it is called a geometric series.

If
$$|r| < 1$$
, it converges to $\frac{a}{1-r}$

If $|r| \geq 1$, it diverges.



Telescoping Sums

With a telescoping sums problem, we write out terms of the series.

We try to write our s_n .

$$\begin{aligned} \mathbf{FX.} \quad s_n &= \sum_{i=1}^n \frac{1}{i(i+1)} \\ &= \sum_{i=1}^n \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) \\ &+ \dots + \left(\frac{1}{n} - \frac{1}{n}\right) + \frac{1}{n+1} \\ &= 1 + \frac{1}{n+1}, \end{aligned}$$

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + 3k + 2}$$

Harmonic and p-Series

There is the special series called the harmonic series.

 $\sum_{n=1}^{\infty} \frac{1}{n}$

This always diverges!

More generally:

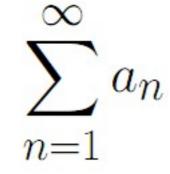
We have the p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

If p > 1, If $p \le 1$, it converges. it diverges.

The First Test: Test for Divergence

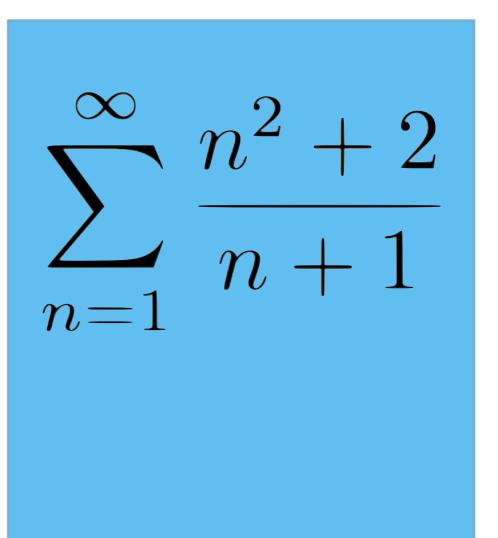
Given a series:



$$\lim_{n \to \infty} a_n \neq 0$$

then, the series diverges.

If: $\lim_{n \to \infty} a_n = 0$ then, NO INFO!



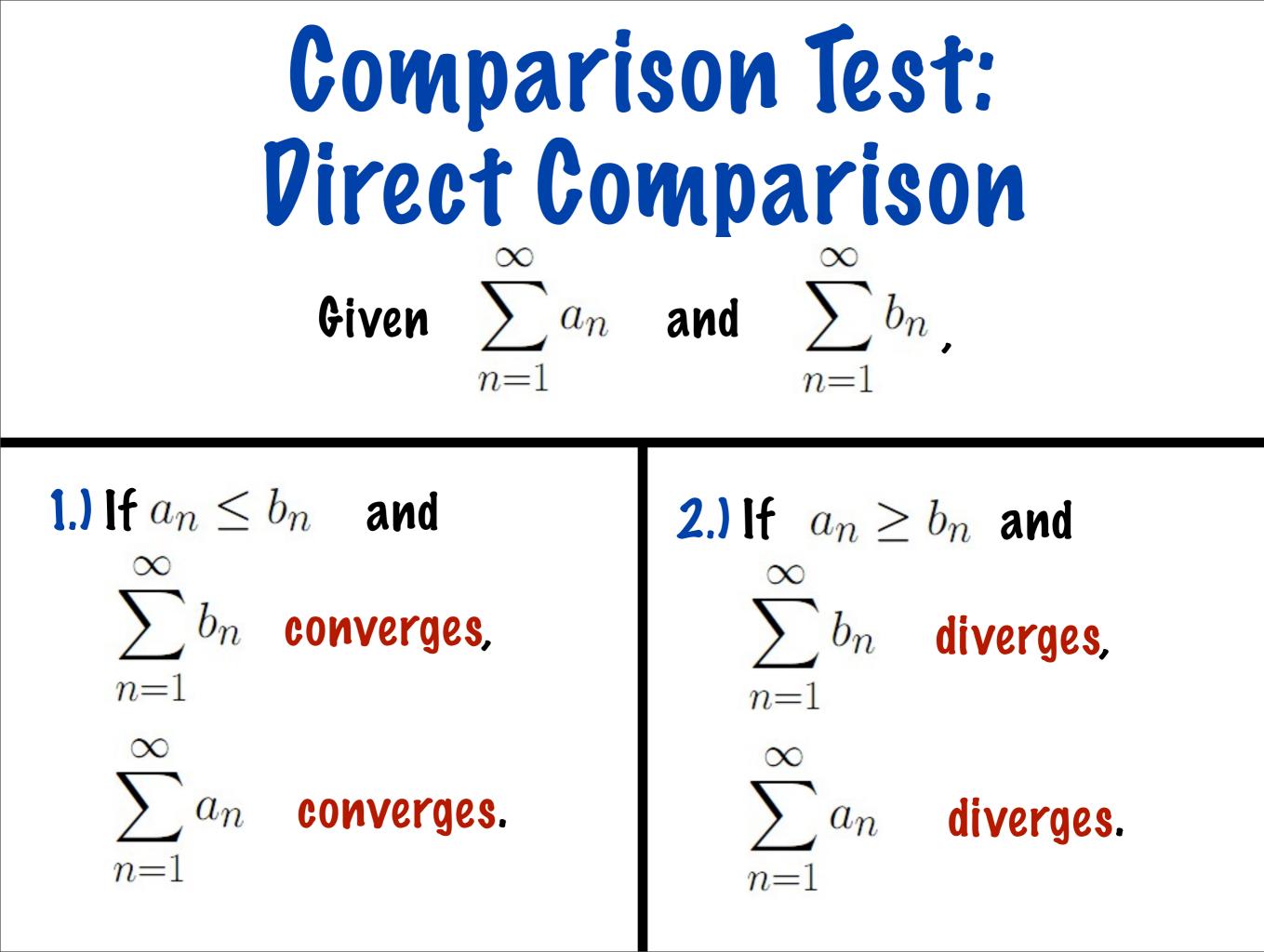
Integral Test

If you are given a series and:

- f(x) is positive over $[c, \infty)$.
- f(x) is continuous over $[c, \infty)$.
- f(x) is decreasing over $[N, \infty)$, where $N \ge c$.

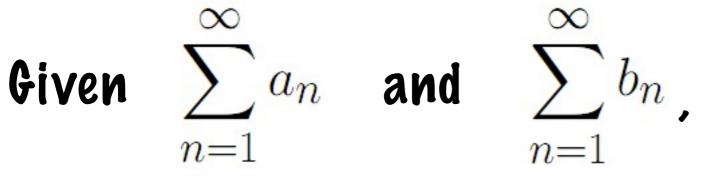
If $\int_{c}^{\infty} f(x) dx$ converges/ diverges, then, $\sum_{n=c}^{\infty} a_n$ also converges or also diverges.

$$\sum_{n=0}^{\infty} \frac{2}{(2+n)^{3/2}}$$



Comparison Test: Limit Comparison Test

 $\sum_{n=1}^{\infty} \frac{n+2}{n^2-1}$



If
$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$
 for c finite and positive,

then both series converge or diverge together.

Comparison Test Examples

 2^n $3^{n} - 1$ $\ln n$ n=2

Alternating Series Test

An alternating series has the form:

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

The series converges if:

$$b_{n+1} \leq b_n$$
 for all n
and
 $\lim b_n = 0$

 $n \rightarrow \infty$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$

Absolute/Conditional Convergence

A series converges absolutely if:



A series converges conditionally if:

$\sum_{n=1}^{\infty} a_n$	converges, but
$\sum_{n=1}^{\infty} a_n $	diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

Ratio Test \sum_{α}^{∞}

Given a series:

If:
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$$

if:
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$$

it diverges.

If:
$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L = 1$$

NO INFO!

$$\sum_{\substack{n=2}}^{\infty} \frac{1}{ne^n}$$

Root Test

 ∞

n=2

 e^{3k}

 $\sum_{k=1}^{n} \frac{c}{k^{3k}}$

Given a series:

If:
$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1$$

it converges.
If: $\lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1$
it diverges.

 ∞

n=1

 a_n

If:
$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = L = 1$$

NO INFO!

Strategies for Series

Always try the Test for Divergence first!

- 1. Does it look like one of the 4 special cases?
 - →Geometric
 - →Telescoping
 - →Harmonic

→p-Series

2.Does it look integrable?

→Use the Integral Test

3.Does it look similar to another series?

→Use one of the Comparison tests

Strategies for Series

- 4. Poes it have $(-1)^n$?
 - → Use Alternating Series Test

5. Does it have n! or an n exponent?

→ Use Ratio/Root Tests

Power Series

A power series centered about a is a series with the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

0

c_n is called the coefficient of the series. Our goal for power series is to:

1. Find the radius of convergence.

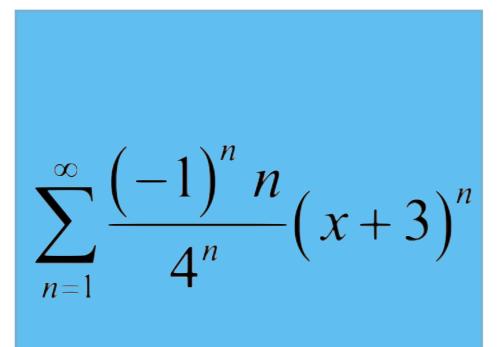
2. Find the interval of convergence.

We use the ratio/root tests

to find the radius and interval of convergence.



Find the radius and interval of convergence.



 $\sum n!(2x+1)^n$ n=0

Representation of Functions

We can represent functions as power series.

Our strategy is to make our functions look like this function.

We will use

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

We can do this by taking the derivative or integral of the series.

to write functions as power series.

Representation of Functions Examples

Write the functions as a power series. Find the radius and interval of convergence.

$g(x) = \frac{1}{\left(1 - x\right)^2}$	

$$h(x) = \ln(5-x)$$

Taylor Series

A Taylor series is a special kind of power series.

t is
$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

with
$$c_n = \frac{f^{(n)}(a)}{n!}$$
.

The n-th degree Taylor polynomial is

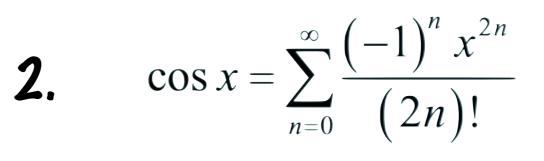
$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i$$

Maclaurin Series

A Maclaurin series is a special kind of Taylor series.

Three Special Maclaurin Series:





3. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Taylor/Maclaurin Series Examples

Find the Taylor series.

$$f(x) = x^4 e^{-3x^2} \text{ about } x = 0$$

$$f(x) = \ln(x)$$
 about $x = 2$

