Sequences and Series Review

Math 1132Q/1152Q

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What is a sequence?

A **sequence** is a list of numbers with a specified order.

Ex. $\{a_1, a_2, a_3, \ldots, a_n\}$

The sequence **converges** if:

$\lim_{n\to\infty} a_n$ exists.

Otherwise, it **diverges**.

$$\left\{\frac{1}{2^k}\right\}_{k=0}^{\infty}$$

What is a series?

A series is the sum of an infinite sequence.

$$\sum_{n=1}^{\infty} a_n$$

A partial sum is the sum:

$$\sum_{i=1}^{n} a_i = s_n$$

If:

$\lim_{n\to\infty} s_n$ exists,

then, the series converges.

Otherwise, it diverges.

Geometric Series

When we have a series of the form:

$$\sum_{n=1}^{\infty} ar^{n-1}$$

it is called a **geometric series.**

If $|r| < 1$,

it **converges** to $\frac{a}{1-r}$

If $|r| \geq 1$, it **diverges.**

$$\sum_{n=1}^{\infty} e^{n}$$

Telescoping Sums

With a telescoping sums problem, we write out terms of the series.

We try to write our s_n.

Ex.

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i\,(i+1)} \\ &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= 1 + \cancel{\left(-\tfrac{1}{2} + \tfrac{1}{2} \right)} + \cancel{\left(-\tfrac{1}{3} + \tfrac{1}{3} \right)} \\ &\quad + \ldots + \cancel{\left(\tfrac{1}{n} - \tfrac{1}{n} \right)} + \tfrac{1}{n+1} \\ &= 1 + \tfrac{1}{n+1}, \end{aligned}$$

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + 3k + 2}$$

Harmonic and p-Series

There is the special series called the harmonic series.

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

This always diverges!

More generally:

We have the p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

If $p > 1$, it converges.

If $p \leq 1$, it diverges.

The First Test: Test for Divergence

Given a series:

$$\sum_{n=1}^{\infty} a_n$$

If: $$\lim_{n\to\infty} a_n \neq 0$$

then, the series diverges.

If: $$\lim_{n\to\infty} a_n = 0$$

then, NO INFO!

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n+1}$$

Integral Test

If you are given a series and:

- $f(x)$ is positive over $[c, \infty)$.
- $f(x)$ is continuous over $[c, \infty)$.
- $f(x)$ is decreasing over $[N, \infty)$, where $N \geq c$.

If $\int_c^\infty f(x)\,dx$ converges/diverges,

then, $\sum_{n=c}^\infty a_n$ also converges or also diverges.

$$\sum_{n=0}^{\infty} \frac{2}{(2+n)^{3/2}}$$

Comparison Test: Direct Comparison

Given $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$,

1.) If $a_n \leq b_n$ and

$\sum_{n=1}^{\infty} b_n$ converges,

$\sum_{n=1}^{\infty} a_n$ converges.

2.) If $a_n \geq b_n$ and

$\sum_{n=1}^{\infty} b_n$ diverges,

$\sum_{n=1}^{\infty} a_n$ diverges.

Comparison Test: Limit Comparison Test

Given $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$,

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ for c finite and positive,

then both series converge or diverge together.

$$\sum_{n=2}^{\infty} \frac{n+2}{n^2-1}$$

Comparison Test Examples

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n - 1}$$

Alternating Series Test

An alternating series has the form:

$$\sum_{n=1}^{\infty} (-1)^n b_n$$

The series converges if:

$$b_{n+1} \leq b_n \text{ for all } n$$

and

$$\lim_{n \to \infty} b_n = 0$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$$

Absolute/Conditional Convergence

A series converges **absolutely** if:

$$\sum_{n=1}^{\infty} |a_n| \quad \text{converges.}$$

A series converges **conditionally** if:

$$\sum_{n=1}^{\infty} a_n \quad \text{converges, but}$$

$$\sum_{n=1}^{\infty} |a_n| \quad \text{diverges.}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$$

Ratio Test

Given a series: $$\sum_{n=1}^{\infty} a_n$$

If: $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$$

it converges.

If: $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$$

it diverges.

If: $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$$

NO INFO!

$$\sum_{n=2}^{\infty} \frac{1}{ne^n}$$

Root Test

Given a series: $$\sum_{n=1}^{\infty} a_n$$

If: $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$$

it converges.

If: $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$$

it diverges.

If: $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$$

NO INFO!

$$\sum_{n=2}^{\infty} \frac{e^{3k}}{k^{3k}}$$

Strategies for Series

Always try the Test for Divergence first!

1. Does it look like one of the 4 special cases?
   - ➡Geometric
   - ➡Telescoping
   - ➡Harmonic
   - ➡p-Series

2. Does it look integrable?
   - ➡Use the Integral Test

3. Does it look similar to another series?
   - ➡Use one of the Comparison tests

Strategies for Series

4. Does it have $(-1)^n$?

   ➡ Use Alternating Series Test

5. Does it have $n!$ or an n exponent?

   ➡ Use Ratio/Root Tests

Power Series

A power series centered about a is a series with the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

c_n is called the coefficient of the series.

Our goal for power series is to:

1. Find the radius of convergence.

2. Find the interval of convergence.

We use the

ratio/root tests

to find the radius and interval of convergence.

Power Series Examples

Find the radius and interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n$$

$$\sum_{n=0}^{\infty} n!(2x+1)^n$$

Representation of Functions

We can represent functions as power series.

We will use

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

to write functions as power series.

Our strategy is to make our functions look like this function.

We can do this by taking the derivative or integral of the series.

Representation of Functions Examples

Write the functions as a power series. Find the radius and interval of convergence.

$$g(x) = \frac{1}{(1-x)^2}$$

$$h(x) = \ln(5-x)$$

Taylor Series

A **Taylor series** is a special kind of power series.

It is $$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$$

with $$c_n = \frac{f^{(n)}(a)}{n!}.$$

The **n-th degree Taylor polynomial** is

$$T_n(x)=\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Maclarin Series

A **Maclaurin series** is a special kind of Taylor series.

It is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

Three Special Maclaurin Series:

1. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2. $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

3. $$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Taylor/Maclaurin Series Examples

Find the Taylor series.

$$f(x) = x^4 \mathbf{e}^{-3x^2} \text{ about } x = 0$$

$$f(x) = \ln(x) \text{ about } x = 2$$

Questions?